# ETMAG

CORONALECTURE 4

Limits of functions

April 6, 12:15

So far we have introduced and investigated the idea of the limit in case of sequences. Sequences are functions with the argument $n$ in $\mathbb{N}$, which means we study limits of rather specific functions and only when the argument approaches infinity. We will now generalize the idea to all real functions and we allow the argument to approach any value, infinite or not.

Take any convergent sequence $(a_n)$ with $\lim_{n\to\infty} a_n = L$ and ask what happens if we apply some function $f$ to all term of the sequence?

Obviously we get a new sequence, $f(a_n)$ which may be convergent to some number P or divergent to + or – infinity, or just divergent, depending on the function. So, this question is more about the function $f$ than about the sequence $(a_n)$.

**Definition.** (Heine)
A number L is the *limit of a function f at x=c* (or *as x approaches c*), iff for every sequence $(x_n)$ convergent to c and such that for every n, $x_n \neq c$, the sequence $f(x_n)$ is convergent to L.

In a more formal way:

$$\lim_{x \to c} f(x) = L \equiv$$

$$(\forall (x_n) \in (\mathbb{R} \setminus \{c\})^{\mathbb{N}}) \left( \lim_{n\to\infty} x_n = c \;\Rightarrow\; \lim_{n\to\infty} f(x_n) = L \right)$$

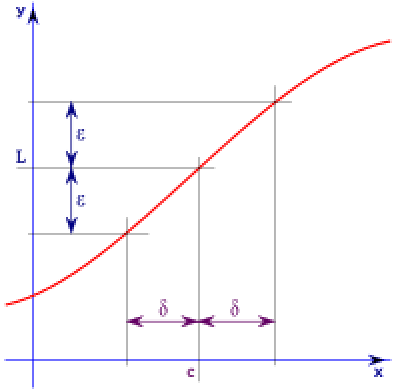
Look at the expression $(\forall (x_n) \in (\mathbb{R} \setminus \{c\})^{\mathbb{N}})$. The "$\forall (x_n)$" part means "for every sequence $(x_n)$" , $(x_n)$ is the symbol we use for sequence consisting of terms $x_1, x_2, \ldots$ etc.

The "$(\mathbb{R} \setminus \{c\})^{\mathbb{N}}$" part means we only consider sequences whose terms are all different from c.

**Definition.** (Augustin-Louise Cauchy)
Let f be a real function and let $c, L \in \mathbb{R}$. We say that L is the *limit of* f, *as x approaches c*, iff

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(\ 0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon\ ).$$

We write then $\lim\limits_{x \to c} f(x) = L$.




Notice that the part "$0<|\ x-c\ |$" means we do not require that $c \in Dom(f)$. In fact we are NOT interested in the value of f for x=c, we do not even assume it exists.
This definition of the limit is called $\varepsilon-\delta$ (Cauchy) definition as opposed to the sequence (Heine) definition.

Picture from Wikipedia

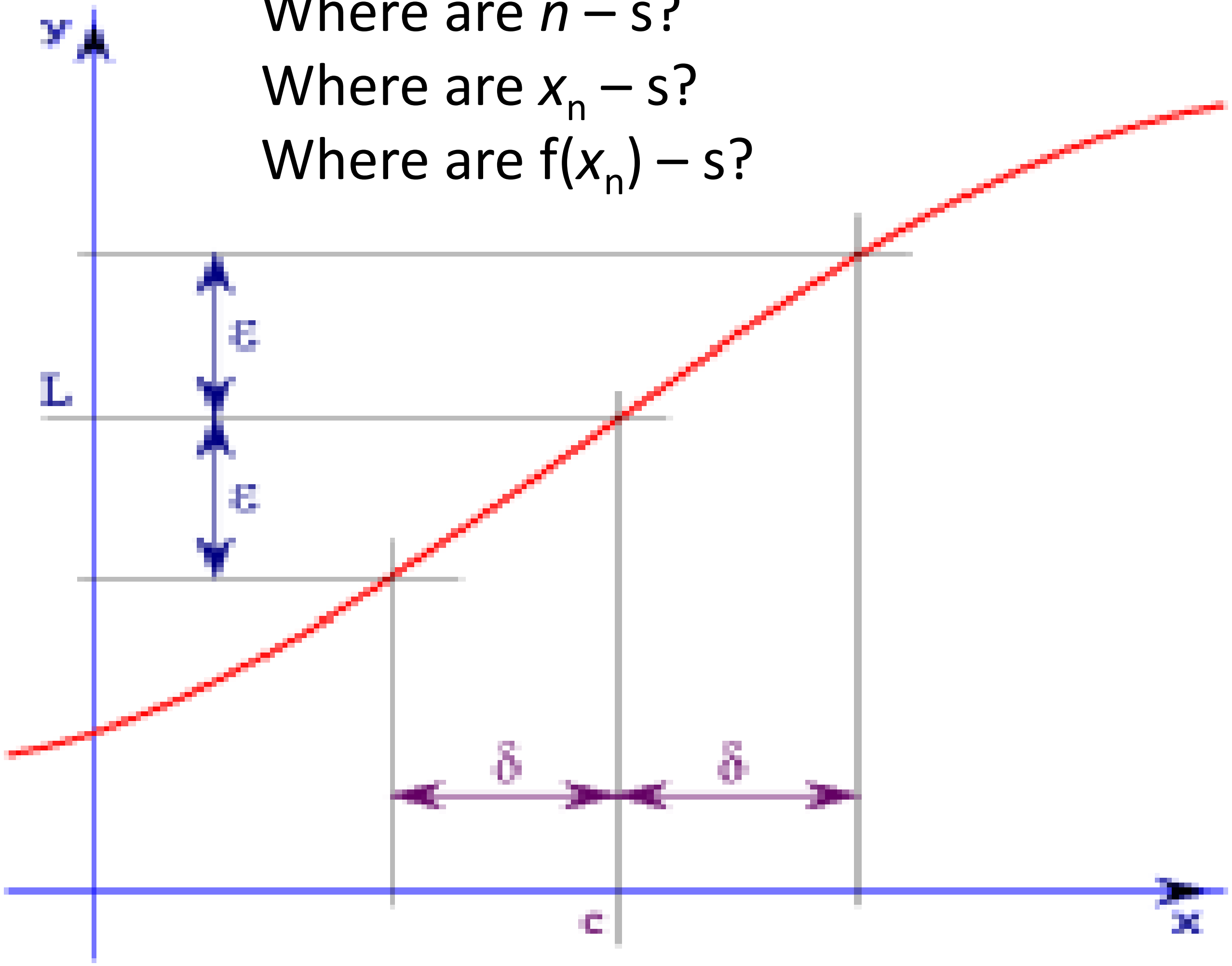
**FAQ.**
Can we illustrate graphically the Heine definition?

**Answer.**

No.

But let's try anyway.

Where are $n$ – s?
Where are $x_n$ – s?
Where are f($x_n$) – s?



**Theorem.**
The two definitions of the limit of a function are equivalent i.e., a number L is the limit of f at some point c in the sense of Cauchy iff the number is the limit of f at c in the sense of Heine.

**Bonus question.** Fully correct and original proof will earn you 3 points, a proof suspected of being somebody else’s minus one point, an honest attempt showing some understanding of the task one point. Submit your answers not later than Tuesday before 6pm.

**Example 1.** Find $\lim\limits_{x\to 0} \sin\frac{1}{x}$ if it exists.

We use Heine definition:

First let $x_n = \frac{1}{2n\pi+\frac{\pi}{2}}$. Then, $\lim\limits_{n\to\infty} \sin x_n = \lim\limits_{n\to\infty} \sin\left(2n\pi + \frac{\pi}{2}\right) = \lim\limits_{n\to\infty} 1 = 1$.

Now use another sequence, $a_n = \frac{1}{2n\pi-\frac{\pi}{2}}$. In this case , $\lim\limits_{n\to\infty} \sin a_n = \lim\limits_{n\to\infty} \sin\left(2n\pi - \frac{\pi}{2}\right) = \lim\limits_{n\to\infty}(-1) = -1$.

We conclude that the function $\sin\frac{1}{x}$ has no limit at 0 because we have found two sequences convergent to 0, $(x_n)$ and $(a_n)$, and the limits of corresponding sequences of values of f differ.

**Example 2.** Find $\lim_{x\to 0} x^2 \sin\frac{1}{x}$, if it exists.

We will use Cauchy definition to show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.
Let ε be a positive real number. We must find a positive number δ such that $0 < |x - 0| < \delta$ implies $\left|x^2 \sin\frac{1}{x} - 0\right| < \varepsilon$. Since for every $x \neq 0$ $\left|\sin\frac{1}{x}\right| \leq 1$ we have $\left|x^2 \sin\frac{1}{x}\right| \leq x^2$. So, if we choose $\delta = \sqrt{\varepsilon}$ we will get $\left|x^2 \sin\frac{1}{x}\right| \leq x^2 < (\sqrt{\varepsilon})^2 = \varepsilon$ as required.

Notice that we are not bothered by the fact that $\sin\frac{1}{x}$ is undefined for x=0

**FAQ.**

1. In example 1, how to find such two sequences?
   There is no useful answer to this question other than the usual, boring one – practice, practice, practice …
2. How do I know whether I should try to show that the function has or that it does not have a limit at a point?
   The same boring answer.
3. Does example 2 indicate that we should try to find some sort of rule which assigns a *delta* to an *epsilon*?
   Yes, exactly. It does not have to be a function, though. Any $\delta$ such that $0 < \delta \leq \sqrt{\varepsilon}$ will do.
4. Sometimes you say *number* sometimes *point*. What is the difference?
   None, in this context. Every point on the real axis is a number and every number is a point on the real axis.

## Limits involving infinity

1. Limits at infinity.

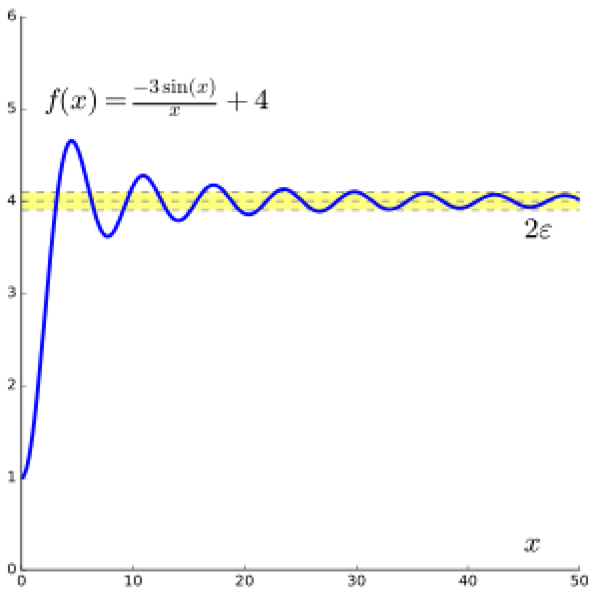
**Definition.**

For a function f(x), a number L is the *limit of* f *as x approaches infinity* iff

(*Cauchy*) $(\forall\varepsilon > 0)(\exists c \in \mathbb{R})(\forall x \in \mathbb{R})(x > c \Rightarrow |f(x) - L| < \varepsilon)$

(*Heine*) for every sequence $(x_n)$: $\lim_{n\to\infty} x_n = \infty \Rightarrow \lim_{n\to\infty} f(x_n) = L$

In a similar way we define the limit of f as *x* approaches *minus infinity*.




*The picture from Wikipedia*

**Limits involving infinity**

2. Infinity as the limit.

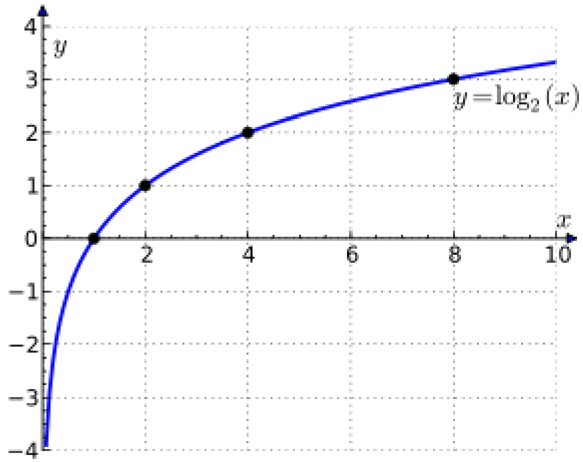
**Definition.**

For a function f(x), and a point c we say $\lim_{x \to c} f(x) = \infty$ iff

(*Cauchy*) $(\forall M \in \mathbb{R})(\exists \delta > 0)(\forall x \in Dom(f))(|x - c| < \delta \Rightarrow f(x) > M)$

(*Heine*) for every sequence $(x_n)$, $\lim_{n \to \infty} x_n = c \Rightarrow \lim_{n \to \infty} f(x_n) = \infty$

In a similar way we define $\lim_{x \to c} f(x) = -\infty$




*The example from Wikipedia.*

$$\lim_{x \to 0^+} \log_2 x = -\infty$$

# We switch to the old presentation here

**Subsequences re-revisited**.

Let $(a_n)$ be a sequence, which means $(a_n)$ is a function $a : \mathbb{N} \to \mathbb{R}$. Take another sequence, say $(b_n)$, i.e. another function $b : \mathbb{N} \to \mathbb{N}$ and **assume that** ***b*** **is increasing**. Then a ∘ b is a subsequence of $a_n$, namely $\left(a_{b_n}\right) = \left(a_{b_1}, a_{b_2}, \dots, a_{b_n}, \dots\right)$ .

If we drop the monotonicity requirement for *b* then we do NOT get, strictly speaking, a subsequence of *a* but we do get a sequence consisting of (some) terms of the original sequence, perhaps only finitely many, perhaps arranged in a messy order.

This means that, from the point of view of the limit, the construction is useless, unless … Unless we assume that the sequence *b* diverges to ∞.

**Theorem.** (Subsequence theorem generalized)
Let $a = (a_n)$ be a sequence. Then $\lim_{n\to\infty} a(n) = L$ iff for every sequence $b=(b_n)$ such that $\lim_{n\to\infty} b(n) = \infty$, we have $\lim_{n\to\infty} (a \circ b)(n) = L$.

**Proof.** ($\Leftarrow$) **Comprehension test.**

($\Rightarrow$) **Challenging comprehension test.** Fully correct and original proof gives you 3 points, a proof suspected of being somebody else's - minus one point, an honest attempt showing some understanding of the task - one point. Submit your answers not later than Tuesday 6pm.

If we temporarily introduce a silly symbol $x(\infty) = \lim_{n\to\infty} x$ for every sequence $x = (x_n)$, convergent, or divergent to $\infty$, then our theorem can be written as, still silly looking,

$$\lim_{n\to\infty} (a \circ b)(n) = a(b(\infty)).$$

One can think about this theorem in term of properties of limits of sequences. It looks like “the limit of a composition of two sequences is the ‘composition of limits’” – with the additional condition that the limit of the “inner” sequence is $\infty$.